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STATIONARY SPECTRAL MEASURES AND THEIR APPLICATIONS[1]

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ERRATA SHEET for Technical Report 102

Page	Line	Reads	Should Read
2	9*	$((,)$	$(,)$
3	8	in separable	in <u>a</u> separable
5	4	$G \times G$	$G_0 \times G$
5	11*	vis	viz
5	11*	y	g
6	6	$\sum_1^n L_2^{N_i}(G,\nu_i)$	$\sum_1^n \oplus L_2^{N_i}(G,\nu_i)$
11	1*	corrolary	corollary
12	1	Corrollary	Corollary
17	12*	$b(t) = \chi_t$ can be chosen	$b(t)$ can be chosen
17	8*	$a(t,\lambda) = b^{-1}(\lambda)b(\lambda + 1)$	$a(t,\lambda) = b^{-1}(\lambda)b(\lambda + t)$
20	7*	It follows	It is known
35	11	Forelli [ ]	Forelli [2]
37	Ref [10]	Mathematiques,	Mathematiques XIV 1965

1. Introduction.

The Von-Neumann Theorem [18] and its generalization [12] have played a significant role not only in quantum mechanics but in prediction and scattering theory. Algebraically, all these problems come under the general structure of an operator-theoretic relation $U_t E(\sigma) U_{-t} = E(\sigma + t)$ between a group representation $\{U_t\}$ and a spectral measure E defined on the Borel subsets of the group. When the group representation is continuous, this problem of the structure of E was studied by G. Mackey [12]. In some applications, however, the representation may not be continuous. In fact $\{U_t\}$ may be a (not necessarily continuous) representation of a subgroup. For example, such a situation arises in the study of analytic measures on a locally compact group [2], [14]. This raised the question of generalizing Mackey's result to the above situation. This is carried out in Section 3.

In the study of "spectral theory" of stationary processes the structure of the spectrum of $\{U_t\}$ plays an important role. In case $\{U_t\}$ is a continuous representation, the basic structure of the spectrum has been given in [7], [8]. It falls in so called continuous case. In their fundamental work H. Helson and D. Lowdenslager [5] studied the "extreme discontinuous" case. The methods used by them are analytic and seem to be totally different from the geometric methods which are used in classical prediction problem. In Section 4, we study the case which falls between these two investigations. In fact, we study the case when $\{U_t\}$ is almost continuous in the sense of K. deLeeuw and I. Glicksberg [10]. Our method is based mainly on the geometry of Hilbert space and the main tool is the decomposition of [10]. Our work in this sense is similar in nature to the classical one. However, the connection between multiplicity and the dimension of process found in [8] fails even in this special case. This can be seen from [15] where (not necessarily continuous) processes of multiplicity one are considered. We study here an extension of this work and at the same time the most general structure of the problem.

The rest of the paper is devoted to applications of these results. As special cases we obtain the main result of scattering theory [9], [17]. Also we obtain the main theorems of [11] and [2]. Section 6 contains new results on the linear prediction of almost continuous processes.

2. Notation and Terminology.

We shall denote by G a separable locally compact abelian group with the group operation denoted by $+$. $\mathcal{B}(G)$ will be the class of Borel subsets of G with μ being the Haar measure on G. $L_2(G,\mu)$ is the Hilbert space of all square integrable functions on G with respect to μ. For each t in G, U_t denotes a unitary operator from a separable Hilbert space H onto itself. $\{U_t,\ t \text{ in } G\}$ is called a (weakly, and hence strongly) continuous representation of G, if the function $(U_t x,y)$ for all x,y in H is a continuous function in t, where $(\ ,\ )$ denotes the inner-product in H. By a spectral measure E, we mean a function E defined on $\mathcal{B}(G)$ with values in the class of orthogonal projections in H. The spectral measure E is called G_0-stationary if

$$(2.1)\qquad U_t\, E(\sigma)\, U_{-t} = E(\sigma + t)$$

for all t in G_0 and $\sigma \in \mathcal{B}(G)$, where G_0 is a subgroup of G.

The spectral measures satisfying (2.1) have applications in the determination of the structure of analytic measures (see [2], [14]). In the case $G = G_0$ and $\{U_t\}$ is a continuous representation one obtains the well-known extension by Mackey [12] of the Stone-Von Neumann Theorem. The word stationary is motivated by its application to prediction theory problems [7] and invariant subspaces (see [9], [6]). As stated before in Section 1, we shall apply the results on spectral measures satisfying (2.1) to partially extend the known results in the above mentioned problems. In order to study the structure of G_0-stationary spectral measures we now require some ideas of the theory of multiplicity in separable Hilbert-space.

Let E be a spectral measure on G. For each element f in H, the measure ρ_f defined by $\rho_f(\sigma) = ||E(\sigma)f||^2$ is called E-measure corresponding to f. The family of all finite measures on $(G, \mathcal{B}(G))$ is divided into equivalence classes by the relation of mutual absolute continuity. If ρ is used to denote the equivalence class to which the measure ρ_f belongs, then ρ will be called the spectral type of f with respect to E. ρ is also referred to as the spectral type belonging to E. We say that the spectral type ρ dominates the spectral type σ $(\rho > \sigma, \sigma < \rho)$ if any (and thus every) measure belonging to σ is absolutely continuous with respect to any measure on ρ. ρ is called Haar type if every measure in ρ is equivalent (mutually absolutely continuous) with respect to μ. ρ and σ are said to be independent spectral types if for any spectral type ν, $\nu < \rho$ and $\nu < \sigma$ implies $\nu = 0$. An element f is said to be of maximal type ρ (with respect to E) if for every g in H, $\rho_g << \rho_f$. It is a known result of this theory that there exists an element f of maximal type with respect to E if H is separable. The subspace $\mathcal{M}_f$ of H generated by $\{E(\sigma)f, \ \sigma \text{ in } \mathcal{B}(G)\}$ is called the cyclic subspace with respect to E generated by f. If $\mathcal{M}_f = H$ then f is called a cyclic or a generating element, and E is called a cyclic spectral measure. For any f, $E(\sigma)$

is reduced by $\mathcal{M}_f$ for each σ in $\mathcal{B}(G)$, we may denote by E_f the restriction of E to $\mathcal{M}_f$. Then E_f is called a cyclic part of E of type ρ, the spectral type of f. Two cyclic parts E_{f_1} and E_{f_2} are orthogonal if $\mathcal{M}_{f_1} \perp \mathcal{M}_{f_2}$. A system of mutually orthogonal cyclic parts of E of type ρ which cannot be enlarged by adding to it more orthogonal cyclic parts is called a maximal system of type ρ. It is a well-known result of this theory that all maximal systems of type ρ have the same cardinal number. This uniquely determined number is defined to be E-multiplicity (for short, multiplicity) of the type ρ. The following theorem of Hellinger-Hahn is the main theorem of this theory.

Theorem 2.1: If H is a separable Hilbert-space with E a spectral measure, then

(i) $H = \sum_1^n \oplus H_i$, where $H_i = \sum_1^{N_i} \oplus \mathcal{M}_{f_{ik}}$;

(ii) for each i, f_{ik} $(k = 1, 2, \ldots, N_i)$ have the same spectral type ρ_i with multiplicity N_i;

(iii) $\rho_1, \rho_2, \ldots, \rho_n$ are mutually independent spectral types;

(iv) $\max(N_1, N_2, \ldots, N_n)$ is called the multiplicity of E;

(v) the couples $(\rho_1, N_1), \ldots, (\rho_n, N_n)$ are unitary invariants corresponding to E; i.e., if $E'(\sigma) = V^{-1}EV$ for V a unitary operator, then $\rho_1, \ldots, \rho_n$ and $N_1, \ldots, N_n$ are the same for E'.

(vi) for each (i,k), $\mathcal{M}_{f_{ik}} = \{\int_G f(t)\, E(dt)\, f_{ik},\ f \in L_2(\rho_{f_{ij}})\}$, where $L_2(\rho_{f_{ik}})$ is the Hilbert-space of square integrable complex-valued functions on G and the integral is called the "stochastic integral" with respect to the measure $\{E(\sigma)f_{ik},\ \sigma \in \mathcal{B}(G)\}$.

We shall call $\{f_{ik}\}$ $(i = 1,\ldots,n,\ k = 1,\ldots,N_i)$ the complete set of generating elements.

Before we conclude this section we introduce the following concepts. We shall call a $M \times M$ matrix-valued function on $G \times G$ a cocycle if

(2.2) (a) $A(s,t)A^*(s,t) = I$ for t in G, s in G_0;

(b) For every fixed s in G_0, the elements of the matrix $A(s,\cdot)$ are $\mathcal{B}(G)$-measurable functions;

(c) $A(s + u,t) = A(s,t)A(u,t + s)$ for each $s,u \in G_0$.

A cocycle is a coboundary if

$$(2.3) \qquad A(s,t) = B(s + t)\, B^{-1}(t)$$

for some $M \times M$ matrix-valued function B on G with measurable entries.

We remark here that the relations (i) and (iii) of Theorem 2.1 and (2.3) hold almost everywhere in t with respect to a fixed measure ν on G which will be specified in each case. Also, in general $M \leq \aleph_0$.

Finally, if χ is a character of G then (χ,g), χ_g, $\chi(g)$ will denote one and the same thing, vis the value of χ at y.

3. Structure of G_0-stationary Measures.

In this section we obtain first the structure of spectral measures E satisfying (2.1). As its consequence we derive Mackey's extension of the Stone-Von-Neuman Theorem. With an additional condition on U_g we then obtain a form of deLeeuw-Glicksberg (see [10], Corr. 5.7) decomposition of a unitary (not necessarily continuous) representation. The section is concluded with an example of a measure which is of type (2.1) and yet does not have the decomposition stated above.

To start with we state some preliminary results which follow as a consequence of Theorem 2.1. In the sequel, we denote for each $N \leq \aleph_0$,

$L_2^N(G,\nu)$ as follows:

$$L_2^N(G,\nu) = \{f_1, f_2, \dots, f_N) = \underline{f}, \quad f_i \in L_2(G,\nu), \quad \sum_1^M \int_G |f_i|^2 d\nu, \text{ finite}\}.$$

On $L_2^N(G,\nu)$ addition and scalar multiplication is defined elementwise and if we define $(\underline{f},\underline{g}) = \sum_1^N \int_G f_i \bar{g}_i d\nu$ then $L_2^N(G,\nu)$ is a Hilbert-space.

<u>Lemma 3.1</u>: Let E be a spectral measure on $\mathcal{B}(G)$. Then there exists an isometry S from H onto $\sum_1^n L_2^{N_i}(G,\nu_i)$ such that $S\,E(\sigma)\,S^{-1}\underline{f} = I_\sigma \underline{f}$ where $I_\sigma(t) = 1$ or 0 according as $t \in \sigma$ or $t \notin \sigma$, and $\nu_i = \rho_{f_{ik}}$ for each i, $k = 1,\dots,N_i$.

<u>Proof</u>: Since for each i, $\rho_{f_{ik}}$ in Theorem 2.1 satisfy $\rho_{f_{i1}} \equiv \rho_{f_{i2}} \equiv \dots \equiv \rho_{f_{iN_i}}$. From this it follows that we can take $\rho_{f_{ik}} = \nu_i$ $(k = 1,\dots,N_i)$. Define now $S\,E(\sigma)f_{ik} = I_\sigma(\cdot)$ for each i and $\sigma \in \mathcal{B}(G)$. Then S can be extended by linearity to an isometry from H_i onto $L_2^{N_i}(G,\nu_i)$, since for each k, $\mathcal{M}_{f_{ik}}$ is isomorphic to $L_2(G,\nu_i)$. The theorem now follows since

$$E(\sigma)\,S^{-1}\underline{f} = \left(\int_\sigma f_1\, E(du) f_{i1}, \dots, \int_\sigma f_{N_i} E(du) f_{iN_i} \right)$$

$$S\,E(\sigma)\,S^{-1}\underline{f} = I_\sigma \underline{f} \qquad \text{for } \underline{f} \in L_2^{N_i}(G,\nu_i)$$

and one can obviously extend this isometry to $\sum_1^n \oplus H_i$. The following result is obvious.

<u>Lemma 3.2</u>: Let $E_t(\sigma) = E(\sigma + t)$ for $t \in G_0$ and $\sigma \in \mathcal{B}(G)$. Then E_t is a spectral measure on $\mathcal{B}(\sigma)$ with $\rho^t_{f_{ik}} \equiv \nu_i^t$ $(k = 1,\dots,N_i)$ for each i, where $\nu_i^t(\sigma) = \nu_i(\sigma + t)$ for $\sigma \in \mathcal{B}(G)$.

We want to remark here that in both the above lemmas we can take $N_1 < N_2 < \dots < N_n$ without loss of generality. In the next lemma we shall use the concept of a quasi-invariant measure. A measure ν on $\mathcal{B}(\sigma)$ is called G_0-quasi-invariant if $\nu(\sigma) = 0$ implies that $\nu(\sigma + t) = 0$ for all t in G_0.

Lemma 3.3: Let E be a G_0-stationary measure given by (2.1). Let $(\nu_1,N_1),\ldots,(\nu_n,N_n)$ be as in Lemma 3.1. Then ν_i, for each i, is G_0-quasi-invariant.

Proof: Since for each t in G_0, $U_t\,E(\sigma)\,U_{-t} = E(\sigma + t)$ and U_t is unitary we obtain that the spectral measure $E(\sigma)$ is unitarily equivalent to the spectral measure $E_t(\sigma) = E(\sigma + t)$ and hence by Theorem 2.1 (v) and Lemma 3.2 $\rho^t_{f_{ik}}$ and $\rho_{f_{ik}}$ for i,k belong to the same spectral type, i.e., $\rho^t_{f_{ik}} \equiv \rho_{f_{ik}}$. Now as noted in the proof of Lemma 3.1 we have $\rho_{f_{ik}} = \nu_i$ $(k = 1,\ldots,N_i)$ for each i. Hence $\nu^t_i \equiv \nu_i$; i.e., ν_i is G_0-quasi-invariant for $i = 1, 2,\ldots,n$. q.e.d.

By equation (2.1) it also follows that $\{U_t f_{ik}\}$ $(k = 1,\ldots,N_i,$ $i = 1,\ldots,n)$ are the complete set of generating elements of H.

Before we proceed to the statement of our main theorem on stationary spectral measures, we motivate the structure of such measures by using the case $n = 1$ and $N_i = 1$ in Theorem 2.1. Let ν be a G_0-quasi-invariant measure on $(G,\mathcal{B}(G))$ then one may consider $H = L_2(G,\mu)$; and define $E(\sigma)x = I_\sigma\cdot x$, $U_s x(\cdot) = a(s,\cdot)x(\cdot+s)\sqrt{\frac{d\nu^s}{d\nu}}\,(\cdot)$ for all x in H and $s \in G_0$, where $a(s,t)$ is a cocycle $(M = 1)$ with respect to ν. Then obviously $E(\sigma)$ is G_0-stationary; one uses here $\frac{d\nu^{s+t}}{d\nu}(\cdot) = \frac{d\nu^s}{d\nu}(\cdot)\,\frac{d\nu^t}{d\nu}(\cdot+t)$. In the next lemma we show that this essentially is the most general situation in the case of multiplicity one.

Proposition 3.1: Let E be a G_0-stationary measure on $\mathcal{B}(G)$ of multiplicity one. Then there exists a G_0-quasi-invariant measure ν on G and a cocycle $a(s,t)$, $(s \in G_0,$ $t \in G)$ with respect to ν such that H is isometric to $L_2(G,\nu)$ and if S denotes this isometry then

(i) $\quad S\,E(\sigma)\,S^{-1}f = I_\sigma\cdot f \qquad$ for all $f \in L_2(G,\mu)$;

(ii) $S\, U_t\, S^{-1} f = a(t,\cdot) f(\cdot + t) \sqrt{\frac{d\nu^t}{d\nu}}(\cdot)$ for t in G_0;

(iii) The function $a(t,\cdot)$ is unique in the sense that if $a'(t,\cdot)$ is any other function then $a'(t,\lambda) a(t,\lambda) = c(\lambda+t)\bar{c}(\lambda)$, a.e. $[\nu]$ for each t in G_0 where c is a measurable function such that $|c| = 1$ a.e. $[\nu]$.

Proof: Let x denote the generating element of E. Then by Lemma 3.1 and 3.3 we get, with $\rho_x = \nu$, that ν is G_0-quasi-invariant and that there exists an isometry S from H onto $L_2(G,\nu)$ such that $S\, E(\sigma)\, S^{-1} f = I_\sigma f$ for all $f \in L_2(G,\nu)$ and $\sigma \in \mathcal{B}(G)$. Let us define $z(\sigma) = E(\sigma)x$ and $z_{-t}(\sigma) = U_t z(\sigma - t)$ for t in G_0. Since $z_{-t}(\sigma) \in E(\sigma)H$ by G_0-stationarity, we obtain that $z_{-t}(\sigma) = \int_\sigma B(t,\lambda) z(d\lambda)$. However

$$||z_{-t}(\sigma)|| = \nu^{-t}(\sigma) = \int_\sigma \frac{d\nu^{-t}}{d\nu}(\lambda)\, \nu(d\lambda).$$

Hence

$$B(t,\lambda) = a(t,\lambda) \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda) \quad \text{a.e. } [\nu], \tag{3.1}$$

where for each t in G_0, $a(t,\lambda)$ is $\mathcal{B}(G)$-measurable and $|a(t,\lambda)|^2 = 1$ a.e. $[\nu]$ giving (a) and (b) of (2.2). Thus one obtains that

$$z_{-t}(\sigma) = E(\sigma)\, U_t x = \int_\sigma a(t,\lambda) \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda)\, z(d\lambda);$$

or, equivalently

$$U_t\, z(\sigma) = E(\sigma + t)\, U_t x = \int_{\sigma+t} a(t,\lambda) \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda)\, z(d\lambda) \tag{3.2}$$

$$= \int_G a(t,\lambda) \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda)\, I_\sigma(\lambda - t)\, z(d\lambda).$$

This implies $S\, U_t\, S^{-1}\, I_\sigma(\lambda) = a(t,\lambda) \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda)\, I_\sigma(\lambda - t)$ and in general

(3.3) $$S\,U_t\,S^{-1}f = a(t,\lambda)\sqrt{\frac{d\nu^{-t}}{d\nu}(\lambda)}\,f(\lambda-t)$$

for t in G_0 and $f \in L_2(G,\nu)$. In order to complete the proof of (i) and (ii) we observe that the property (c) of (2.2) of the cocycle now follows from (3.3) and

$$\frac{d\nu^{-t_1-t_2}}{d\nu}(\lambda) = \frac{d\nu^{-t_1}}{d\nu}(\lambda)\,\frac{d\nu^{-t_2}}{d\nu}(\lambda - t_2)$$

with

(3.4) $$S\,U_{t_1+t_2}S^{-1} = S\,U_{t_1}S^{-1}\,S\,U_{t_2}S^{-1}.$$

To prove (iii) suppose that there is another function $a'(t,\lambda)$ and an isometry S_1 such that (i) and (ii) hold. Then with $z_1(\sigma) = S_1^{-1}I_\sigma$ we get $z_1(\sigma) = \int_\sigma c(u)\,z(du)$ for some $c \in L_2(G,\nu)$. But $||z_1(\sigma)||^2 = \nu(\sigma)$ which gives $c(u) = 1$ a.e. $[\nu]$. Also $U_t\,z_1(\sigma) = \int_\sigma a'(t,\lambda)\,z_1(d\lambda + t)$ [from (3.2)]. Therefore

$$U_t\,z_1(\sigma) = \int_\sigma a'(t,\lambda)\,c(\lambda+t)\,z(d\lambda+t) = \int_\sigma a(t,\lambda)\,c(\lambda)\,z(d\lambda+t).$$

Thus giving $a'(t,\lambda)\,\bar{a}(t,\lambda) = c(\lambda+t)\,\bar{c}(\lambda)$.

q.e.d.

The next proposition essentially brings out the general case. The proof of the next proposition will be given in detail.

<u>Proposition 3.2</u>: Let E be a G_0-stationary spectral measure on $\mathcal{B}(G)$ with a single spectral type ρ of multiplicity N. Then there exists an isometry S from H onto $L_2^N(G,\nu)$ for some quasi-invariant measure ν and a cocycle $A(t,\lambda)$ $(M = N)$ with respect to ν, such that

(i) $S\,E(\sigma)\,S^{-1}\underline{f}(\cdot) = I_\sigma(\cdot)\,\underline{f}(\cdot)$ for $\underline{f} \in L_2^N(G,\nu)$;

(ii) $S\,U_t\,S^{-1}\underline{f}(\cdot) = \sqrt{\frac{d\nu^t}{d\nu}}(\cdot)\,A(t,\cdot)\,f(\cdot + t)$ a.e. ν;

(iii) If there exists another $A'(t,\lambda)$, such that it satisfies (ii), then $A'^*(t,\lambda)\, A(t,\lambda) = C(\lambda + t)\, C^*(\lambda)$ where $C(\lambda)\, C^*(\lambda) = I$ a.e. $[\nu]$.

Proof: As observed in Lemma 3.1, one can choose $\rho_{f_1} = \dots = \rho_{f_N} = \nu$ where $f_1,\dots,f_N$ is a complete set of generating elements of H. Furthermore by Lemma 3.3, ν is G_0-quasi-invariant. Also there exists an isometry S [by Lemma 3.1] from H onto $L_2^N(G,\nu)$ such that $S\, E(\sigma)\, S^{-1}\underline{f}(\cdot) = I_\sigma(\cdot)\, \underline{f}(\cdot)$ for $\sigma \in \mathcal{B}(G)$ and $\underline{f} \in L_2^N(G,\nu)$. Denote now by $z_i(\sigma) = E(\sigma) f_i$. Then we have as in proposition 3.1, $U_t\, z_i(\sigma - t) \in E(\sigma)\, H$ by G_0-stationarity of E. Hence

$$U_t\, z_i(\sigma - t) = \sum_{j=1}^{N} \int_\sigma b_{ij}(t,\lambda)\, z_j(d\lambda).$$

However

$$||U_t\, z_i(\sigma - t)||^2 = \nu^{-t}(\sigma) = \int_\sigma \frac{d\nu^{-t}}{d\nu}(\lambda)\, \nu(d\lambda).$$

This implies

$$\sum_{1}^{N} |b_{ij}(t,\lambda)|^2 = \frac{d\nu^{-t}}{d\nu}(\lambda) \quad \text{a.e. } [\nu]. \tag{3.4}$$

Also $U_t\, z_i(\sigma - t) \perp U_t\, z_j(\sigma - t)$ gives

$$\sum_{1}^{N} \int_\sigma b_{ij}(t,\lambda)\, \bar{b}_{kj}(t,\lambda) = \frac{d\nu^{-t}}{d\nu}(\lambda)\, \delta_{ik} = \begin{cases} \dfrac{d\nu^{-t}}{d\nu}(\lambda) & i = k, \\ 0 & i \neq k. \end{cases} \tag{3.5}$$

We thus obtain that for each $t \in G_0$ and $\lambda \in G$ an $N \times N$ matrix such that

$$B(t,\lambda)\, B^*(t,\lambda) = \left\{ \frac{d\nu^{-t}}{d\nu}(\lambda)\, \delta_{ik} \right\} \tag{3.6}$$

where $\{\ \ \}$ is an $N \times N$ matrix. It follows that

$$B(t,\lambda) = A(t,\lambda) \left\{ \sqrt{\frac{d\nu^{-t}}{d\nu}}(\lambda)\, \delta_{ik} \right\}$$

where for each t, $A(t,\lambda)$ is a $N \times N$ matrix valued function with

$\mathcal{B}(G)$-measurable entries and $A(t,\lambda)\, A^*(t,\lambda) = I$ a.e. ν. Denote by $\underline{z}(\sigma) = (E(\sigma)f_1, \ldots, E(\sigma)f_N)^*$. Then, formally, [see (3.2)]

$$U_t\, \underline{z}(\sigma) = (E(\sigma + t)\, U_t f_1,\, E(\sigma + t)\, U_t f_2, \ldots, E(\sigma + t)\, U_t f_N)^*,$$

that is

$$U_t\, \underline{z}(\sigma) = \int_\sigma A(t,\lambda) \left\{\sqrt{\frac{d\nu^{-t}}{d\nu}}\,(\lambda)\, I_\sigma(\lambda - t)\, \delta_{ik}\right\} \underline{z}(d\lambda). \tag{3.7}$$

Equation (3.7) gives

$$S\, U_t\, S^{-1}\, \{I_\sigma(\cdot)\, \delta_{ik}\} = A(t,\lambda) \left\{\sqrt{\frac{d\nu^{-t}}{d\nu}}\,(\lambda)\, I_\sigma(\lambda - t)\, \delta_{ik}\right\} \tag{3.8}$$

which implies (ii). Properties (a) and (b) of (2.2) of $A(t,\lambda)$ are verified before. Property (c) of (2.2) of $A(t,\lambda)$ follows as in equation (3.4) of Proposition 3.2. The proof of (iii) being similar to (iii) of Proposition 3.2 is omitted.

The proof of the following theorem now follows from Lemma 3.3 and Proposition 3.2.

<u>Theorem 3.1</u>: Let E be a G_0-stationary spectral measure. Then

(a) There exists mutually singular measures $\nu_1, \nu_2, \ldots, \nu_n$, each ν_i, G_0-quasi-invariant and cardinal numbers $N_1, \ldots, N_n$ $(N_1 < N_2 < \ldots < N_n \le \aleph_0)$;

(b) Cocycles $A_1, \ldots, A_n$ such that for each i A_i is a cocycle $(M = N_i)$ with respect to ν_i;

(c) An isometry S from H onto $\sum_1^n \oplus L_2^{N_i}(G, \nu_i)$;

such that for $\underline{f} \in L_2^{N_i}(G, \nu_i)$

(i) $\quad S\, E(\sigma)\, S^{-1} \underline{f}(\cdot) = I_\sigma(\cdot)\, \underline{f}(\cdot)$;

(ii) $\quad S\, U_t\, S^{-1} \underline{f}(\cdot) = A_i(t,\cdot) \left\{ \dfrac{d\nu_i^{-t}}{d\nu_i}(\cdot)\, \delta_{jk} \right\}_{j,k} \underline{f}(\cdot - t) \qquad t \in G_0$;

(iii) A_i are unique in the sense of Proposition 3.2 (iii).

<u>Remark</u>: The above theorem does not use the topology of G. The following corrolary is of interest.

Corollary 3.1: Let E be a G_0-stationary spectral measure with $(\nu_1,N_1),\ldots,(\nu_n,N_n)$ as in Theorem 3.1. If there exist (σ-finite) measures $m_1,\ldots,m_n$ such that $m_i \equiv \nu_i$ and $m_i(\sigma + t) = m_i(\sigma)$ for all t in G_0, $\sigma \in \mathcal{B}(G)$, then there exist cocycles $A_1,\ldots,A_n$ (each A_i an $N_i \times N_i$ matrix) with respect to $m_1,\ldots,m_n$, respectively, and an isometry V from H onto $\sum_1^n \oplus L_2^{N_i}(G,m_i)$ such that $V\,E(\sigma)\,V^{-1}\underline{f} = I_\sigma \underline{f}$ and $V\,U_t\,V^{-1}\underline{f} = A_i(t,\lambda)\,\underline{f}(\cdot - t)$ for $\underline{f} \in L_2^{N_i}(G,m_i)$.

Proof: Define $S_1\underline{g} = \sqrt{\frac{d\nu}{dm}}\,\underline{g}$. Then S_1 can be extended to an isometry from $\sum_1^n \oplus L_2^{N_i}(G,\nu_i)$ onto $\sum_1^n \oplus L_2^{N_i}(G,m_i)$. Denote now by $V = S_1 S$. Then V is an isometry from H onto $\sum_1^n \oplus L_2^{N_i}(G,m_i)$. Now the result follows from Theorem 3.1 and the fact that for $\underline{f}$ in $L_2^{N_i}(G,m_i)$

$$V\,U_t\,V^{-1}\underline{f} \;=\; S_1 S\,U_y\,S^{-1}S_1^{-1}\underline{f}(\cdot) \;=\; S_1 S\,U_t\,S^{-1}\sqrt{\frac{dm}{d\nu}(\cdot)}\,\underline{f}(\cdot)$$

$$= S_1\,A(t,\cdot)\,\underline{f}(\cdot - t)\sqrt{\frac{dm}{d\nu}(\cdot - t)}\sqrt{\frac{d\nu^t}{d\nu}(\cdot)}$$

$$= A(t,\lambda)\,\underline{f}(\cdot - t)$$

q.e.d.

In general, if $G_0 \neq G$, the existance of equivalent G_0-invariant (σ-finite) measures for a given G_0-quasi-invariant measure is not known. However, for $G_0 = G$ the following lemma of Mackey [12] gives the required result.

Lemma M1: (Lemma 3.3 [12] p. 318). Let G be a separable locally compact group. Let ν be a non-zero regular measure on $\mathcal{B}(G)$ such that G is the union of (at most) countably many measurable subsets of finite ν-measure, and such that every right translate of every set of ν-measure zero is again of ν-measure zero. Then ν is equivalent to right invariant Haar measure μ in G.

In our case, G being abelian, $\nu \equiv \mu$. We now come to the extension of Mackey's theorem (Theorem 1 of [12]) to the case where the representation is not assumed to be continuous. As an immediate consequence of the above lemma and Corollary 3.1 we get that E will have Haar type. In addition, we shall show that the cocycle $A(t,\lambda)$ occurring in Proposition 3.2 is a coboundary. In general, it is known that a cocycle defined on $(G_0 \times G)$ where G_0 is a subgroup of G and $G_0 \neq G$ may not be a coboundary (see [6], p. 259).

Following is an extension of Mackey's Theorem.

Theorem 3.2: Let $\{U_t\}$ be a unitary representation of a group G on a separable Hilbert-space H. Let E be a stationary spectral measure on $\mathcal{B}(G)$ such that

$$U_t\, E(\sigma)\, U_{-t} = E(\sigma + t) \qquad \text{for } t \in G, \quad \sigma \in \mathcal{B}(G); \tag{3.9}$$

then

(i) E has multiplicity N of Haar type $(N \leq \aleph_0)$.

(ii) There exists an isometry V from H onto $L_2^N(G,\mu)$ such that $V\,E(\sigma)\,V^{-1}\underline{f}(\cdot) = I_\sigma(\cdot)\,\underline{f}(\cdot)$ and $V\,U_t\,V^{-1}\underline{f}(\cdot) = A(t,\cdot)\,\underline{f}(\cdot - t)$ for $\underline{f} \in L_2^N(G,\mu)$ where μ is the Haar measure on G and $A(t,\lambda)$ is $N \times N$ cocycle on $G \times G$ with respect to μ.

(iii) $A(t,\lambda)$ is a coboundary.

As observed before, (i) and (ii) follow once we take $G = G_0$. The non-trivial conclusion of Theorem 3.2 is (iii) for which we need the following measure theoretic lemma of Mackey [12].

Lemma M2: (Lemma 3.1 [12], p. 317). Let X_1 and X_2 be two measure spaces with measures ν_1 and ν_2. Suppose that each is a sum of countably many measurable sets of finite measure and that there is a countably generated Borel field $\mathcal{C}$ of measurable subsets of X_2 such that every measurable subset of X_2 differs from some member of $\mathcal{G}$ by a set of measure zero. Let f be a complex-valued function defined on $X_1 \times X_2$ which is measurable and essentially bounded as a function of X_2 for each fixed point in X_1. Suppose that $\int f(x,y)\, I_\sigma(y)\, \nu_2(dy)$ is measurable on X_1 for each measurable subset σ of X_2 of finite measure. Then there exists a function f' measurable on $X_1 \times X_2$ such that for each x in X_1, $f(x,y) = f'(x,y)$ for almost all $y \in X_2$.

Proof of Theorem 3.2: Proof of (i) and (ii) follows from Lemma M1, Proposition 3.2 and Corollary 3.1. We now proceed to prove (iii). Now let $A(t,\lambda)$ be the cocycle as in Proposition 3.2. Then by cocycle property (c) of 2.2

$$(3.10) \qquad A(t+s,\lambda) = A(t,\lambda)\, A(s,\lambda+t) \qquad \text{a.e. } [\mu] \text{ for } t,s \in G.$$

Hence for $t,s \in G$

$$(3.11) \qquad A^{-1}(t,\lambda)\, A(t+s,\lambda)\, A^{-1}(s,\lambda+t) = I \quad \text{a.e. } [\mu]$$

where I is identity matrix. Denoting by $c^*_{ij}(t,s,\lambda)$ the elements, we get that $c^*_{ij}(t,s,\lambda)$ is jointly measurable in (t,λ) over $G \times G$ since from (3.10) $A^{-1}(t,\lambda)\, A(t+s,\lambda)$ is jointly measurable in (t,λ). Also a.e. $[\mu]$ $c^*_{ij}(t,s,\lambda) = \delta_{ij}$ for all (t,s), i.e., $c^*_{ij}(t,s,\lambda) \le 1$. Now let σ be $\mathcal{B}(G) \times \mathcal{B}(G)$ measurable set of finite measure. Then

$$\int_\sigma\!\!\int c_{ij}{}^*(t,s,\lambda)\, d\lambda dt = \delta_{ij}\, \mu \times \mu(\sigma)$$

and hence measurable in s. From Lemma M2 we obtain that there exists a function $c_{ij}(t,s,\lambda)$ jointly measurable in (t,s,λ) such that $c^*_{ij}(t,s,\lambda) = c_{ij}(t,s,\lambda)$. Hence we can replace $A^{-1}(t,\lambda)\, A(t+s,\lambda)\, A^{-1}(s,\lambda+t)$ by a measurable function of (t,s,λ) which we shall assume now. That is

$$(3.11) \qquad A^{-1}(t,\lambda)\, A(t+s,\lambda)\, A^{-1}(s,\lambda+t) = I \qquad \text{a.e. } [\mu \times \mu \times \mu].$$

There exists by Fubini theorem a point $\lambda_0 \in G$ such that

$$(3.12) \qquad A^{-1}(t,\lambda_0)\, A(t+s,\lambda_0)\, A^{-1}(s,\lambda_0+t) = I \quad \text{a.e.} \quad [\mu \times \mu].$$

Putting $\lambda_0 + t = u$,

$$(3.13) \qquad A^{-1}(u-\lambda_0,\lambda_0)\, A(u-\lambda_0+s,\lambda_0)\, A^{-1}(s,u) = I \quad \text{a.e.} \quad [\mu \times \mu],$$

that is

$$A(s,u) = A^{-1}(u-\lambda_0,\lambda_0)\, A(u-\lambda_0+s,s).$$

Define $B(u) = A(u-\lambda_0,\lambda_0)$. . Then

$$(3.14) \qquad A(s,u) = B^{-1}(u)\, B(u+s) \qquad \text{a.e.} \quad [\mu \times \mu].$$

(3.14) given that $A(s,u)$ is a coboundary, thus completing the proof of the theorem.

It is crucial in many problems of analysis to prove that a given cocycle is a coboundary. Previously, (iii) of Theorem 3.2 was proven under the assumption that $A(t,\lambda)$ was jointly measurable in (t,λ). However, we do not make such an assumption. Also from (iii), we can obtain that H is isomorphic to $B^{-1}L_2^N(G,\mu) = \{\underline{f}\, B^{-1};\ \underline{f} \in L_2^N(G,\mu)\}$. $B^{-1}L_2^N(G,\mu)$ is a Hilbert-space if $(\underline{f}\, B^{-1}, \underline{g}\, B^{-1})_{B^{-1}L_2^N(G,\mu)}$ equal $(\underline{f},\underline{g})_{L_2^N(G,\mu)}$. Furthermore, the property that $B^{-1}(\lambda)\, B(\lambda+t)$ is measurable in λ for each fixed t, makes $B^{-1}L_2^N(G,\mu)$ invariant under translations; as

$$\begin{aligned}(\underline{f}\, B^{-1})(\lambda - t) &= \underline{f}(\lambda - t)\, B(\lambda - t)\, B(\lambda)\, B^{-1}(\lambda) \\ &= \underline{g}_t(\lambda)\, B^{-1}(\lambda) \qquad \text{where } \bar{g}_t(\lambda) \in L_2^N(G,\mu).\end{aligned}$$

Hence $\underline{f}(\lambda - t)\, B^{-1}(\lambda - t) \in B^{-1}L_2(G,\mu)$ for each $t \in G$. We have now the following corollary to Theorem 3.2, which will reduce to Mackey's theorem if we put $B^{-1} = I$.

Corollary 3.2: Let E be a stationary spectral measure on G of multiplicity N. Then there exists $N \times N$ unitary-matrix-valued function B and an isometry V_1 from H onto $B^{-1}L_2^N(G,\mu)$ such that

(i) $B^{-1}(\lambda)\, B(\lambda + t)$ is measurable in λ for each fixed t;

(ii) $V_1\, E(\sigma)\, V_1^{-1}\underline{f}(\cdot) = I_\sigma(\cdot)\underline{f}(\cdot)$ for $\underline{f} \in B^{-1}L_2^N(G,\mu)$, $\sigma \in \mathcal{B}(G)$;

(iii) $V_1\, U_t\, V_1^{-1}\underline{f}(\cdot) = \underline{f}(\cdot - t)$ for all $t \in G$ and $\underline{f} \in B^{-1}L_2^N(G,\mu)$;

(iv) B is unique upto multiplication by $N \times N$ unitary matrix valued function with measurable entries.

Before we go to Mackey's extension of Stone-von Neumann Theorem we want to obtain the following theorem under the assumption that the cyclic parts of E are reduced by U_t for all t in G. (It should be noted that, in case $\{U_t\}$ is a continuous representation this assumption is satisfied.) Using the following theorem and a result of deLeeuw and Glicksberg we can prove that this assumption is equivalent to $|(U_t x,y)|$ being continuous in t for each (x,y) in H. Under this assumption we can also exhibit the structure of the spectrum of $\{U_t\}$.

Theorem 3.3: Let $\{U_t\}$ be a unitary representation of G and E be a G-stationary spectral measure on $\mathcal{B}(G)$ with multiplicity N. If there exist N cyclic parts of E which are reduced by $\{U_t\}$ for all t, then we have

(i) There exist at most countable number of (not necessarily continuous) characters $\chi^{(1)},\ldots,\chi^{(k)}$ $(k \le N)$ of G and mutually orthogonal subspaces $H_{\chi^{(1)}},\ldots,H_{\chi^{(k)}}$ such that

(ii) $H = \sum_1^k \oplus H_{\chi^{(i)}}$;

(iii) U_t for each t is reduced by $H_{\chi^{(i)}}$ and if $U_t^{(i)} = \chi^{(i)}(t)U_t^{\chi^{(i)}}$ where $U_t^{\chi^{(i)}}$ denotes the restriction of U_t to $H_{\chi^{(i)}}$, then $U_t^{(i)}$ is a continuous representation of G in terms of the unitary operators on $H_{\chi^{(i)}}$;

(iv) For each x, y in H $t \to |(U_t x,y)|$ is a continuous function on G;

(v) If $U_x = \int_{\hat{G}^d}(t,\gamma)\ \beta(d\gamma)$ denotes the spectral representation of U_t (see F. Riezz and B. Sz-Nagy [16], p. 392) then β lies on the countable number of cosets of $\hat{G}$ in $\hat{G}^d$;

(vi) The function $B(t)$ of Theorem 3.2 can be chosen to be a diagonal matrix with entries $\{\chi_t{}^{(1)},\ldots, \chi_t{}^{(k)}\}$ for almost all χ, where if $k \leqslant N$, some of the entries are repeated.

The proof of the theorem will be given in two stages. The first part, which is the content of the following lemma, is based on the ideas used by M. Nadkarni [15].

<u>Lemma 3.4</u>: Let G be a separable locally compact group and E be a G-stationary spectral measure on $\mathcal{B}(G)$. Assume that E is cyclic. Then

(i) The spectral measure of $\{U_t\}$ sits on a single coset of $\hat{G}$ in $\hat{G}_d$.

(ii) There exists a (not necessarily continuous) character χ of G such that $\{\chi(t)\ U_t\}$ is a continuous representation of G and $t \to |(U_t x,y)|$ is a continuous function on G for each $x,y \in H$.

(iii) $a(t,x)$ is a coboundary and $b(t) = \chi_t$ can be chosen to be χ_t for some fixed χ.

<u>Proof:</u> We know from Theorem 3.2 that H is isomorphic to $L_2(G, \mu)$. Furthermore if $x \in H$ then $x = \int_G \varphi_x(\lambda)\ z(d\lambda)$ $\quad \varphi_x \in L_2(G, \mu)$.

Also $U_t x = \int_G a(t,\lambda)\ \varphi_x(\lambda-t) z(d\lambda)$. But by (3.14) $a(t,\lambda) = b^{-1}(\lambda) b(\lambda+1)$.

Further $b^{-1}(t)b^{-1}(\lambda)b(\lambda+t)$ is jointly measurable in (t,λ) and $\int b^{-1}(t)b^{-1}(\lambda)b(\lambda+t)d\lambda$ is continuous in t for every measurable set σ of σ finite measure. Hence $(b^{-1}(t)U_t x,y) = \int_G b^{-1}(t)b^{-1}(\lambda)b(\lambda+t)\varphi_x(\lambda)\overline{\varphi_y(\lambda)}\ d\mu$ is continuous in t. Hence $|(U_t x,y)|$ is a continuous function on G. Therefore there exists a (not necessarily continuous) character χ (see K. deLeeuw and I. Glicksberg [10], p. 173) such that $H = H_\chi$. The existance of a single character here follows from the assumption of multiplicity one. This completes the proof of (ii). Since $b(t)$ is unique upto multiplication by a measurable function of absolute value one we can take $b(t) = \chi_t$ for each t. Let us now consider $\{\chi_t U_t\}$. It is a group of unitary operators and hence

$$(3.15) \qquad \chi_t \, U_t = \chi_t \int_{\hat{G}_d} (\gamma, t) \, \beta(d\gamma).$$

From the fact that $\{\chi_t U_t\}$ is continuous group of unitary operators we obtain $\chi_t U_t = \int_G (t,u) \, \hat{\beta}(du)$ where $\hat{\beta}$ is the spectral measure on $\hat{G}$; equivalently

$$(3.16) \qquad U_t = \int_{\hat{G}} (t, u - \chi) \, \hat{\beta}(du).$$

From (3.15) and (3.16) we obtain (i). Since (iii) has been obtained before, this completes the proof of the lemma.

q.e.d.

<u>Proof of Theorem 3.3</u>: Proof is now obtained by repeated application of Lemma 3.4, since by assumption there exist cyclic parts of E which reduce U_t. Let $E^{(i)}$ be a cyclic part E which reduces U_t and $U_t{}^{(i)}$ be the restriction of U_t to $\mathcal{M}_f(i)$. Then $U_t{}^{(i)} E^{(i)}(\sigma) \, U_{-t}{}^{(i)} = E^{(i)}(\sigma + t)$ and $E^{(i)}$ is cyclic. This implies, by Lemma 3.4, that there exists a (not necessarily continuous) character $\chi^{(i)}$ such that $\{\chi_t^{(i)} \, U_t^{(i)}\}$ is continuous representation of G on $\mathcal{M}_{f(i)}$. The fact that there exist at most countable number of characters follows from the fact that $N \leq \aleph_0$. For different $\mathcal{M}_f(i)$ the corresponding $\chi^{(i)}$ may not be distinct. Let $\{\chi^{(1)},\ldots,\chi^{(k)}\}$ be the distinct characters. Then obviously $k \leq N$. If $H_{\chi^{(i)}} = \sum_j{}' \oplus \mathcal{M}_{f(j)}$ where the sum Σ' is taken over, those j's for which the corresponding $\mathcal{M}_{f(j)}$ has the property that $\{\chi_t^{(i)} U_t^{(j)}\}$ is continuous. Now (i), (ii) and (iii) are obvious.

Let $x,y \in H$, $x \in H_{\chi^{(i)}}$, $y \in H_{\chi^{(j)}}$. Then $|(U_t x,y)| = 0$ if $i \neq j$ and $|(U_t x,y)| = |(\chi_t^{(i)} U_t^{(i)} x,y)|$ if $i = j$. Hence $t \to |(U_t x,y)|$ is a continuous function on G.(v) follows from the fact that for x in H

$$U_t x = \sum_{i=1}^{k} U_t^{(i)} x_i$$

where $x = \sum_1^k x_i$ and $x_i \in H_{\chi_i}$ and hence

$$(U_t x,y) = \sum_1^k (U_t^{(i)} x_i, y_i)$$

where $y = \sum_1^k y_i$. Now

$$(U_t x,y) = \int_{\hat{G}^d} (t,\gamma)(\beta(d\gamma)x,y) = \sum_1^k \int_{\hat{G}} (t,\gamma - \chi^{(i)})(\hat{\beta}(d\gamma)x,y).$$

This implies that

$$(\beta(\sigma)x,y) = \sum_1^k (\hat{\beta}_i(\sigma \cap \hat{G} + \chi^{(i)})x,y)$$

where $\hat{\beta}_i(\sigma \cap \hat{G} + \chi_i) = \hat{\beta}_i(\sigma \cap \hat{G})$ for all $\sigma \in \mathcal{B}(\Gamma)$, $\hat{G} + \chi^{(i)}$ denoting the coset of $\hat{G}$ in $\hat{G}_d$ corresponding to $\chi^{(i)}$. Hence β lies on the countable number of cosets of $\hat{G}$ in $\hat{G}^d$. The proof of (vi) is as follows.

$A(t,x)$ is diagonal follows from the fact that each $\mathcal{M}_{\zeta_f(i)}$ is reduced by $\{U_t\}$. The entries are discontinuous character is then a consequence of Lemma 3.4 (iii).

q.e.d.

We would like to show that when U_t is strongly continuous, B of Corollary 3.2 can be chosen to be identity matrix. Since B is unique upto multiplication by a $N \times N$ unitary matrix valued measurable function it is enough to show that B can be chosen to be measurable. We have by strong continuity of U_t, $\int_\sigma a_{ij}(t,s)\,ds$ is a continuous function of t for all σ where $a_{ij}(t,s)$ are elements of the matrix

$A(t,s)$. Hence we can choose $a_{ij}(t,s)$ measurable in (t,s). But $B(t)\ B^{-1}(t+s) = A(t,s)$ which implies that $B(t)$ can be chosen to be measurable in t. Hence by Corollary 3.2, H is isomorphic to $L_2^N(G,\mu)$ and if V_1 denotes this isometry then $V_1\ E(\sigma)\ V_1^{-1}\underline{f} = I_\sigma \underline{f}$ and $V_1\ U_t\ V_1^{-1}\underline{f} = \underline{f}(\cdot - t)$. Let H_i be the subspace of H generated by $V^{-1}(0,\ldots,\underset{\substack{\uparrow \\ \text{ith place}}}{f_i},\ldots,0)$. Then H_i is isomorphic to $L_2(G,\mu)$ and since $L_2(G,\mu)$ is invariant under translation operator, H_i is reduced by U_t. We thus have the following

<u>Theorem 3.4</u>: (Theorem 1, Mackey [12], p. 314.) Let G be a separable locally compact abelian group. Let U_t be a continuous representation of G in H. Let E be a spectral measure on $\mathcal{B}(G)$ such that $U_t\ E(\sigma)\ U_{-t} = E(\sigma + t)$. Then H can be represented as a direct sum of at most countably many closed subspaces each of which is invariant under U_t (for all t) and $E(\sigma)$ for all $\sigma \in \mathcal{B}(G)$, and for each of which there is a norm preserving map S onto $L_2(G,\mu)$ such that $S\ U_t\ S^{-1}f(\cdot) = f(\cdot + t)$ and $S\ E(\sigma)\ S^{-1}f = I_\sigma f$.

Observe that S can be taken to be restriction of V_1 to H_i. Mackey's theorem is stated in equivalent form by using the group of unitary operators

$$V_\tau = \int_G (\tau,t)\ E(dt) \qquad \tau \in G. \tag{3.17}$$

It follows that $U_t V_\tau = (\tau,t)\ V_\tau U_t$ is equivalent to $U_t\ E(\sigma) = E(\sigma + t)U_t$.

In conclusion we would like to remark as to the importance of (v) of Theorem 3.3. It enables one to study the Spectral Theory of not necessarily mean continuous processes and also analytic measures. It should be observed that if one looks at the group V_τ of (3.17) one obtains the equation $V_\tau \beta(\Delta) = \beta(\Delta + \tau)\ V_\tau$ where Δ is a Borel subset of $\hat{G}_d$ [14], i.e., the spectral measure β on $\hat{G}_d$ is

G-stationary. Further by (v) of Theorem 3.3 one can show that under the assumption of the theorem that $(\beta(\Delta)x,x)$ is absolutely continuous with respect to $\Sigma\, \mu_{\hat{G}} * \epsilon_{\chi^{(i)}}$ where $\mu_{\hat{G}}$ is the Haar measure on $\hat{G}$ and $\epsilon_{\chi^{(i)}}$ is the "point-measure" at $\chi^{(i)}$ and $*$ denotes the convolution. The following example shows the essentiality of the condition of Theorem 3.3.

Let R_d denote the real line with the discrete topology and $B = \hat{R}_d$ its compact dual. Let φ be the continuous isomorphism of R, the real line with the usual topology, into B. Let μ be a finite measure on $\varphi(R)$ equivalent to the Lebesgue measure on $\varphi(R)$. Let J be the subgroup of rational numbers in R_d and $H \subset B$ be the anihilator of J (i.e., $H = \{x\colon \chi_j(x) = 1 \;\; j \in J\}$). It is easy to see that H is a closed subgroup of B and $H \cap \varphi(R) = \{0\}$. Let ν be a non-atomic regular measure on H such that $L_2(H,\nu)$ is separable. For any Baire set E in B consider the function on H

$$f_E(x) \;=\; \mu(E + x) \qquad\qquad x \in H.$$

It can be shown that $f_E(x)$ Baire measurable on H. Now consider the measure m on B defined by $m(E) = \int_H \mu(E + x)\;\nu(dx)$. Since both $L_2(B,\mu)$ and $L_2(B,\nu)$ are separable, $L_2(B,m)$ is separable. Further m is quasi-invariant under $\varphi(R)$. We see this as follows:

$$\begin{aligned} m(E) \;&=\; 0 \Leftrightarrow \nu\{x\colon \mu(E + x) = 0\} \;=\; 0 \Leftrightarrow \nu(x\colon \mu(E + x + \varphi(t) = 0) \\ &=\; 0 \Leftrightarrow m(E + \varphi(t)) \;=\; 0. \end{aligned}$$

Here the equivalence in the second step is due to quasi-invariance of μ under $\varphi(R)$. Since ν is non-atomic and $H \cap \varphi(R) = \{0\}$, it follows that $m(\varphi(R) + x) = 0$ for all $x \in B$; i.e., m measure of every coset of $\varphi(R)$ is zero. Now as m is quasi-invariant under $\varphi(R)$, by a result of deLeeuw-Glicksberg [11] we can show

$$||m - m_{\varphi(t)}|| \to 0 \quad \text{as} \quad t \to 0,$$

where $m_{\varphi(t)}$ is the measure defined by $m_{\varphi(t)}(E) = m(E + \varphi(t))$ and $|| \ldots ||$ denotes the total variation of $m - m_{\varphi(t)}$. It is easy to see from this that the group V_t of unitary operators in $L_2(B,m)$ defined by $(V_t f)(\cdot) = \frac{dm_{\varphi(t)}}{dm}(\cdot)\, f(\cdot + \varphi(t))$ is strongly continuous so that V_t has spectral representation of the type

$$V_t = \int e^{it\lambda}\, dE_\lambda,$$

where E_λ is a spectral resolution of the identity. Let U_τ be defined in $L_2(B,m)$ as follows:

$$U_\tau f = \chi_\tau f, \qquad \chi_\tau \text{ character on } B \text{ corresponding to } \tau \in R_d.$$

Then it is easy to check that

(i) $V_t\, U_\tau = e^{it\tau}\, U_\tau V_t$;

(ii) $U_\tau\, E(\sigma)\, U_{-\tau} = E(\sigma + \tau)$;

(iii) $(U_\tau f, g) = \int_B \chi_\tau(x)\, f(x)\, \bar{g}(x)\, m(dx)$.

Now no cyclic part E can be reduced by $\{U_\tau\}$. For if there were a cyclic subspace of E reduced by U_τ, the non-trivial part of spectral measure of U_τ would sit on a coset of $\varphi(R)$ in B. But (iii) and the fact that $m(\varphi(R) + x) = 0$ for all $x \in B$, shows that this is impossible.

We would like to express our thanks to Professor R. V. Chacon for suggesting to us the idea of the above example.

In the next section we obtain a converse to Theorem 3.3 using the decomposition due to deLeeuw and Glicksberg [10].

4. Spectrum of an Almost Continuous Representation.

Let $\{U_t\}$ t in G be a representation of G by unitary operators on H, such that $t \to |(U_t x,y)|$ $(x,y \in H)$ is a continuous function on G. Following deLeeuw and Glicksberg we shall refer to these representations as almost continuous. Our purpose in this section is to show that under this condition on $\{U_t\}$, the cyclic parts of the spectral measure E, with $U_t\, E(\sigma)\, U_{-t} = E(\sigma + t)$ $\sigma \in \mathcal{B}(G)$, $t \in G$, are invariant under $\{U_t\}$. This provides a converse to Theorem 3.3. As a consequence of this we can obtain the explicit form of the spectrum of U_t.

Theorem 4.1: Let $\{U_t\}$ be an almost continuous representation of G by unitary operators in a separable Hilbert-space H. Denote by E a spectral measure on $\mathcal{B}(G)$ where for each $\sigma \in \mathcal{B}(G)$, $E(\sigma)$ is a projection operator in H and

$$(4.1) \qquad U_t\, E(\sigma)\, U_{-t} = E(\sigma + t) \qquad \sigma \in \mathcal{B}(G), \quad t \in G.$$

Then $E = E_1(\sigma) + \ldots + E_N(\sigma)$, each $E_i(\sigma)$ is cyclic of Haar type and $H = H_1 + \ldots + H_N$, each H_i is cyclic of Haar type and $E_i(\sigma_1)E_j(\sigma_2) = 0$ if $i \neq j$ or $\sigma_1 \cap \sigma_2 = \emptyset$ and $\{U_t\}$ reduces $E_i(\sigma)$. Here N denotes the multiplicity of Haar type with respect to E.

The proof of this theorem depends on Corollary 5.7, p. 173 of deLeeuw and Glicksberg [10].

Proof: It follows from the Corollary that there exist discontinuous characters $\{\chi_1, \ldots, \chi_n\}$ of G such that

$$(4.2) \qquad H = H_c \oplus \sum_{1}^{n} \oplus H_{\chi_i}$$

where H_c and H_{χ_i} $(i = 1,2,\ldots,n)$ are reduced by $\{U_t\}$ and $U_t^{\,c}$, the restriction of $\{U_t\}$ to H_c, is a continuous representation of G.

Also if $U_t^{\chi_i}$ denotes the restriction of U_t to H_{χ_i}, then $(t,\chi_i)\, U_t^{\chi_i} = U_t^{(i)}$ is a continuous representation of G. In fact $H_{\chi_i} = \{x | x \in H,\ ((t,\chi_i)U_t x,y)$ is a continuous function on G for each y in $H\}$.

Let $\sigma \in \mathcal{B}(G)$. Since by (4.1)

$$
\begin{aligned}
(4.3) \qquad ((t,\chi_i)U_t\, E(\sigma)x,y) &= (E(\sigma + t)(t,\chi_i)\, U_t x,y) \\
&= ((t,\chi_i)\, U_t x, E(\sigma + t)y),
\end{aligned}
$$

we obtain that H_{χ_i} reduces $E(\sigma)$ for all σ and H_c reduces E. Let $\hat{E}_c$ and $\hat{E}_i$ denote the restriction of E to H_c and H_{χ_i}. Then for $t \in G$ and $\sigma \in \mathcal{B}(G)$

$$
\begin{aligned}
(4.4) \qquad U_t^{(c)}\, \hat{E}_c(\sigma)\, U_t^{(c)} &= \hat{E}_c(\sigma + t) \quad \text{and} \\
U_t^{(i)}\, \hat{E}_i(\sigma)\, U_t^{(i)} &= \hat{E}_i(\sigma + t).
\end{aligned}
$$

We can now take H_c and H_{χ_i} $(i = 1,\ldots,n)$ as separable Hilbert-space and $\hat{E}_c$ and $\hat{E}_i$ as G-stationary measures on them. Since in (4.4) $\{U_t^{(c)}\}$ and $\{U_t^{(i)}\}$, for each i, is a continuous representation of G by unitary operators of H_c and H_{χ_i} for each i, respectively by Theorem 3.4, we obtain that $\hat{E}_c = \hat{E}_c^{(1)} +\ldots+ \hat{E}_c^{(M_c)}$ and $\hat{E}_i = \hat{E}_i^{(1)} +\ldots+ \hat{E}_i^{(M_i)}$ such that $\{U_t^{(c)}\}$ is reduced by the cyclic subspaces $\mathfrak{S}\{\hat{E}_c^{(k)}(\sigma)f_c^{(k)},\ \sigma \in \mathcal{B}(G)\}$ $(k = 1,\ldots,M_c)$ and by $\mathfrak{S}\{\hat{E}_i^{(j)}(\sigma)f_i^{(j)},\ \sigma \in \mathcal{B}(G)\}$ $(j = 1,\ldots,M_k)$. Also each $f_c^{(k)}$ and $f_i^{(j)}$ are of Haar type. Hence by renumbering

$$
E = E_1(\sigma) +\ldots+ E_N(\sigma)
$$

where $N = M_c + M_1 +\ldots+ M_n$ and $E_i(\sigma)$ obviously have Haar type and are cyclic. In order to complete the proof it suffices to prove that N is the multiplicity. The following argument follows the argument used by G. Kallianpur and V. Mandrekar (see [7], p. 632).

Let $\{E_\beta(\sigma)\}_{\beta=1}^{N_0}$ be another orthogonal system of Haar type consisting of cyclic parts of E. Here the type being Haar type is a consequence of (4.1) and Theorem 3.2. We want to prove $N_0 \le N$. By separability of H both N_0 and N cannot exceed $\aleph_0$. Hence if $N = \aleph_0$, there is nothing to prove. Assume now that $N < \aleph_0$ and N finite. Let us rename f's by $g_1, \ldots, g_N$ and let h_β be the generating element of E_β. Since the type is the same for both the systems we can suppose, without loss of generality, that $\nu = \rho_{g_i} = \rho_{h_\beta}$ for all i and β. Now

$$h_\beta = \sum_{k=1}^{N} \int_G F_{ik}(u)\, E(du)\, g_i$$

where $\sum_1^N \int_G |F_{ik}(u)|^2\, \nu(du) < \infty$. For every $\sigma \in \mathcal{B}(G)$,

$$(E(\sigma)\, h_\beta, h_\gamma) = \int_\sigma \sum_{i=1}^{N} F_{i\beta}(u)\, F_{i\gamma}(u)\, \nu(du).$$

We therefore have for $u \notin C_{\beta\gamma}$, $\nu(C_{\beta\gamma}) = 0$

$$\sum_1^N F_{i\beta}(u)\, F_{i\gamma}(u) = \delta_{\beta\gamma} = \begin{cases} 0 & \text{if } \beta \neq \gamma \\ 1 & \text{if } \beta = \gamma \end{cases} \quad \text{for each } \beta,\gamma. \tag{4.5}$$

Let $C = \bigcup_{\beta,\gamma} C_{\beta\gamma}$. Then $\nu(C) = 0$ and $u_0 \notin C$.

$$\sum_1^N F_{i\beta}(u_0)\, F_{i\gamma}(u_0) = \delta_{\beta\gamma}. \tag{4.6}$$

Now let $a_\beta = \{F_{1\beta}(u_0), \ldots, F_{N\beta}(u_0)\}$. Then equation (4.6) implies that a_β are N_0 vectors orthogonal in N-dimensional space contradicting $N < N_0$. Hence $H_0 \le N$, i.e., N is the multiplicity of Haar type with respect to E.

q.e.d.

Now as in (3.17) let us define for each $\tau \in \hat{G}$, $V_\tau = \int_{\hat{G}} (\tau,u)\, E(du)$. Then V_τ by Stone's theorem is a continuous representation of $\hat{G}$ by unitary operators on H. We also know that $U_t = \int_{\hat{G}_d} (t,u)\, \beta(du)$. Further, under the assumption that $t \to |(U_t x,y)|$ is continuous, we obtain that β lies on a countable number of cosets of $\hat{G}$ in $\hat{G}_d$; i.e., for all $x,y \in H$, $(\beta(\Delta)x,y) = \nu_i^{xy}(\Delta)$ where $\nu_i(\Delta) = \sum_{i=1}^{n} \nu_i(\Delta \cap (G + \chi_i))$, $n \leq \aleph_0$. This means that there exists a measure ν on $\hat{G}$ such that $\nu_i^{xy}(\Delta) = \nu([\Delta - \chi_i] \cap \hat{G})$. In fact, let β_i be the restriction of β to H_{χ_i}. Then $\beta_i(\Delta) \neq 0$ only if $\Delta \subset \hat{G} + \chi_i$. Since V_τ is also reduced by H_{χ_i} (E is reduced!) we have for $\tau \in \hat{G}$, and $\Delta \subset \hat{G} + \chi_i$,

$$V_\tau^{(i)} \beta_i(\Delta) = \beta_i(\Delta + \tau)\, V_\tau^{(i)}, \tag{4.7}$$

where $V_\tau^{(i)}$ is the restriction of V_τ to H_{χ_i}. Using Theorem 3.1 we therefore get that

$$H_{\chi_i} = \sum_{\alpha=1}^{M_i} \oplus H_{\chi_i}^{\alpha}$$

such that each $H_{\chi_i}^{\alpha}$ is isomorphic to $L_2(G + \chi_i, \mu_{\hat{G}} * \epsilon_{\chi_i})$ where $\mu_{\hat{G}}$ is the linear measure on $\hat{G}$, $\epsilon_{\chi_i}(\Delta) = \begin{cases} 0 & \text{if } \chi_i \notin \Delta \\ 1 & \text{if } \chi_i \in \Delta \end{cases}$, since $\mu_{\hat{G}} * \epsilon_{\chi_i}$ is the $\hat{G}$-quasi-invariant measure on $\hat{G} + \chi_i$ where $\hat{G}$ is identified as a subgroup of $\hat{G} + \chi_i$. We thus have the following

<u>Theorem 4.2</u>: If $\{U_t\}$ is an almost continuous representation of G by unitary operators on H and $\{V_\tau\}_{\tau \in G}$ is as defined in (3.16) and satisfies $U_t V_\tau = (\tau,t)\, V_\tau U_t$, then

$$H = \sum_{i=1}^{n} \sum_{\alpha=1}^{M_i} \oplus H_{\chi_i}^{(\alpha)}$$

such that for each i, $H_{\chi_i}^{(\alpha)}$ is isomorphic to $L_2(\hat{G} + \chi_i, \mu_{\hat{G}} * \epsilon_{\chi_i})$ for $\alpha = 1,2,\ldots,M_i$ where $\{\chi_1,\ldots,\chi_n\}$ are (not necessarily continuous) characters of G.

Theorems 3.3, 4.1 and 4.2 give the complete structure of the spectrum of the groups of unitary operators $\{U_t\}_{t \in G}$ and $\{V_\tau\}_{\tau \in G}$ satisfying the operator equation

$$(4.8) \qquad U_t V_\tau = (\tau, t) V_\tau U_t$$

on a separable Hilbert-space under the assumption that $\{V_\tau\}$ is a continuous representation of G and $\{U_t\}$ is almost continuous representation of G. In the work of Lax-Phillips and Sinai the assumption on $\{U_t\}$ is that it is a strongly continuous representation. We shall extend these results to $\{U_t\}$ almost continuous and apply to the study of prediction and representation of (almost continuous) stationary processes and the study of analytic measures on groups with ordered duals. Since these problems are more specific than the general problems treated in Sections 3 and 4, it is not out of place to introduce them in themselves and therefore we divide the rest of the paper in three sections each dealing with each one of the problems stated above.

5. Invariant Subspaces.

Let Γ be "ordered;" i.e., there is a continuous homomorphism ψ from Γ into R, the real line with usual topology. Let H be a separable Hilbert-space and $\{U_\gamma, \gamma \in \Gamma\}$ be a group of unitary operators on H. $\mathcal{M}$ is called an invariant subspace of H if $U_\gamma \mathcal{M} \subset \mathcal{M}$ for $\psi(\gamma) < 0$. It is called "outgoing" following Lax and Phillips [9] if it is invariant and satisfies

$$(5.1) \qquad \bigcap_{\psi(\gamma)<0} U_\gamma \mathcal{M} = \{0\} \quad \text{and} \quad \bigvee_\gamma U_\gamma \mathcal{M} = H.$$

It is known that "outgoing" subspaces play an important role in the prediction theory and scattering theory (see [6], [13], [9]).

Let us now define H_t to be the subspace of H generated by $\{U_\gamma x, x \in \mathcal{M}, \psi(\gamma) \le t\}$ and and $E(t)$, the projection onto H_t.

Then E is clearly a resolution of the identity andlet E denote the spectral measure on R generated by it. Furthermore $U_{\gamma_0} H_t = H_{t+\psi(\gamma_0)}$. We thus have

$$(5.2) \qquad U_{\gamma_0} E(\sigma) = E(\sigma + \psi(\gamma_0))\, U_{\gamma_0}$$

for $\gamma_0 \in \Gamma$ and $\sigma \in \mathcal{B}(R)$. Let R_0 denote the subgroup $\psi(\Gamma)$ of R. Then (5.2) gives

$$(5.3) \qquad U_{\gamma_0} E(\sigma)\, U_{-\gamma_0} = E(\sigma + \psi(\gamma_0)).$$

From (5.3) and a method very similar to that of Theorem 3.1, we obtain the following extension of Lax-Phillips-Sinai theorem, [9].

Theorem 5.1: If $\mathcal{M}$ is an "outgoing" subspace of a representation $\{U_\gamma\}$ $\gamma \in \Gamma$ of an "ordered" group Γ by unitary operators on a separable Hilbert-space H, then.

(i) There exist mutually singular measures $\nu_1, \nu_2, \ldots, \nu_n$ on R, each ν_i R_0-quasi-invariant, and matrix functions $A_1, \ldots, A_n$ on $\Gamma \times R$ such that for each i, A_i is a cocycle $(M = N_i)$ with respect to ν_i;

(ii) An isometry S from H to $\sum_1^n \oplus L_2^{N_i}(R, \nu_i)$ such that $S\, U_\gamma\, S^{-1} \underline{f}$ $= A_i(\gamma, \cdot) \{\frac{d\nu_i^{-\psi(\gamma)}}{d\nu_i}(\cdot)\, \delta_{jk}\}_{j,k}\, \underline{f}(\cdot - \psi(\gamma))$ for $\gamma \in \Gamma$ and $\underline{f} \in L_2^{N_i}(R, \nu_i)$;

(iii) $S\mathcal{M} = \sum_1^n \oplus L_2^{N_i}(R, \nu_i; (-\infty, 0])$ where $L_2^{N_i}(R, \nu_i; (-\infty, 0]) =$ $\{\underline{f},\ \underline{f} \in L_2^{N_i}(R, \nu_i),\ \underline{f}(\lambda) = 0,\ \lambda > 0\}$.

If $\Gamma = R_d$ we get the following theorem which is an extension of Lax-Phillips-Sinai theorem to (not necessarily) continuous case as a consequence of Corollary 3.2.

Theorem 5.2: Let $\{U_t\}_{t \in R}$ be a (not necessarily continuous) representation of R by unitary operators on a separable Hilbert-space H. If $\mathcal{M}$ is an "outgoing" subspace of H with respect to $\{U_t\}$ then there exists an isometry V_1 from H onto $B^{-1}L_2^N(R,\mu)$ where B is an $N \times N$ matrix-valued function, N the multiplicity and μ the Lebesgue measure. Further

(i) $B^{-1}(\lambda)\, B(\lambda + t)$ is a Borel measurable function of λ for each t;

(ii) $V_1\, U_t\, V_1^{-1}\underline{f} = \underline{f}(\cdot - t)$ for $t \in R$ and $\underline{f} \in B^{-1}L_2^N(R,\mu)$;

(iii) $V_1\mathcal{M} = B^{-1}L_2^N(R,\mu;(-\infty,0))$ where $B^{-1}L_2^N(R,\mu;(-\infty,0]) =$

$\{\underline{f};\ \underline{f} \in B^{-1}L_2^N(R,\mu),\ \underline{f}(\lambda) = 0 \text{ for } \lambda > 0\}$.

If $\{U_t\}$ is strongly continuous one can choose B^{-1} to be measurable as in Theorem 3.3 and obtain

Corollary 5.1: (Lax-Phillips; Theorem 1, [9]) Let $\mathcal{M}$ be an "outgoing" subspace for a strongly continuous representation $\{U_t\}$ of R on a separable Hilbert-space H. Then H can be represented isometrically as $L_2^N(R,\mu)$ (N being multiplicity of E), U_t going into translation and $\mathcal{M}$ in the space of all functions with support on negative reals.

6. Almost Continuous Stationary Processes.

Let $(\Omega,\mathcal{F},P)$ be a probability space. For each t real denote by x_t a complex-valued Borel measurable function $(\Omega,\mathcal{F},P)$. $\{x_t\}$ is called stochastic process. $\{x_t\}$ is called second order stationary (henceforth, stationary) if $\int |\underline{x}_t|^2\, dP$ is finite and $\int x_t \bar{x}_s\, dP = r(t - s)$ for $s < t$. r is called the covariance function of x_t. The study of univariate and multivariate processes have been a subject of several papers (see for references [7]). Classically, the processes indexed by R are studied for the case when $\int |x_t - x_s|^2\, dP \to 0$ as $s \to t$. These are called mean continuous processes. One can regard the above problem as a geometric one by considering the Hilbert-space $L_2(x)$, the subspace of $L_2(\Omega,P)$ generated by $\{x_\tau, \tau \in R\}$ and the unitary group U_t, given for each t, by $U_t x_s = x_{s+t}$, and extended to $L_2(x)$ by linearity. In the mean continuous case, $\{U_t\}$ becomes a strongly continuous representation of R. A s.o. process is called purely non-deterministic if $\bigcap_t L_2(x;t) = \{0\}$, where $L_2(x;t)$ is the subspace of $L_2(x)$ generated by $\{x_\tau, \tau \le t\}$.

For purely non-deterministic mean continuous stationary processes the study of prediction and representation was done by O. Hanner [3] and was extended through its connection with R-stationary spectral measures to mean-continuous multidimensional processes in [7]. In this section we remove the assumption of mean continuity. We shall assume that $|r(t)|$ is a continuous function of t. In the abstract geometric problem this gives that the representation $\{U_t\}$ defined above is almost continuous. This case is in between the classical case of mean continuity and the "extreme discontinuous" case studied in [6]. The interest of this condition on r lies in the fact that the complete analytic criterion for pure non-determinism can be given using classical work, [1], as opposed to the new analysis used by Helson and Lowdenslager in [6].

Let $E(t)$ be the projection of $L_2(x)$ onto $L_2(x;t)$. If $\{x_t\}$ is purely non-deterministic we obtain that $E(t)$ is a resolution of the identity. By stationarity, $U_t L_2(x;s) = L_2(x;s + t)$, which gives $U_t\, E(s)\, U_{-t} = E(s + t)$. Denoting by E the spectral measure generated by $\{E(t)\}$, we get that E is R-stationary. Furthermore, as remarked before, $\{U_t\}$ is an almost continuous representation of R. Hence we have the following lemma as a consequence of Theorem 4.2.

<u>Lemma 6.1</u>: If $\{x_t\}$ is a second-order purely non-deterministic stationary stochastic process with $|r(t)|$ continuous in t, then

(i) $x_t = x_t^{\chi_1} + \ldots + x_t^{\chi_n}$ where $\{\chi_1, \ldots, \chi_n\}$ are (not necessarily continuous) distinct characters of R and $\{x_t^{\chi_i}\}$ are mutually orthogonal;

(ii) $(\chi_i, t)\, x_t^{\chi_i}$ are mean continuous stationary processes and hence the spectral measure ν_i of $x_t^{\chi_i}$ given by $\int x_t^{\chi_i} \bar{x}_s^{\chi_i}\, dP = \int \chi_t(\lambda)\, \nu_i(d\lambda)$ lies on the coset $R + \chi_i$ on R into $\hat{R}_d$;

(iii) The spectral measure ν_i is absolutely continuous with respect to the linear measure ℓ_{χ_i} on $R + \chi_i$.

Now using the classical criterion for pure non-determinism (see J. Doob [1], p. 586) we get

<u>Theorem 6.1</u>: Let $\{x_t\}$ be a stationary stochastic process with covariance function r such that $|r(t)|$ is continuous. Then $\{x_t\}$ is purely-non-deterministic if and only if for all $i = 1, \ldots, k$

$$\int_{R+\chi_i} C_{\chi_i}(\lambda) \log \frac{d\nu_i}{d_{\chi_i}}(\lambda)\, \mu_{\chi_i}(d\lambda) > -\infty$$

where $C_{\chi_i}(\lambda)$ is the density of Cauchy measure on the coset of $R + \chi_i$ with respect to linear measure on $R + \chi_i$.

7. <u>Analytic Measures</u>.

In this section, we study the properties of analytic measures. Classically one studies the analytic measures μ on T^2 such that $\hat{\mu}(m,n) = 0$ for $m+\lambda n \leq 0$ for a fixed irrational λ. Recently, we have used the method of stationary spectral measures to obtain quasi-invariance of analytic measures on compact groups. At the same time a similar method was used by F. Forelli [2] for obtaining some generalizations of some work of Rudin and Stout. He studies measures μ on T^k with $\mu(n_1, \ldots n_k) = 0$

if $n_i < 0$ for $i = 1, 2, \ldots, k$. We shall give here a general result about such measures on arbitrary compact groups. We would like to express our thanks to Professor I. Glicksberg for suggesting to us the general format and Professor F. Forelli for making available to us the preprint of his paper [2].

Let G be a compact abelian group and Γ its dual. Let Q be a subsemigroup of Γ and Γ_0 a subgroup of Γ such that $\Gamma = \Gamma_0 + Q$. Let ψ be a continuous homomorphism from Γ_0 to R such that $\psi(\gamma) \leq 0$ implies $\gamma + Q \subset Q$. Let μ be a regular Borel signed measure on G and $|\mu|$ its total variation measure. Assume $\hat{\mu}(u) = 0$ for $u \in Q$. Let ρ denote the natural map from G onto $G/\Gamma_0^{\perp}$, $\Gamma_0^{\perp}$ being the annihilator of Γ_0 in G and φ; $R \to G/\Gamma_0^{\perp}$ the dual map of ψ.

Let $e(\cdot) = \frac{d\mu}{d|\mu|}(\cdot)$ and $\mathcal{M}_s$ be the subspace of $L_2(|\mu|)$ generated by $\{e(\cdot)(\gamma_0 + u); \gamma_0 \in \Gamma_0, \ \psi(\gamma_0) \leq s\}$. The following properties of the subspaces $\mathcal{M}_s$ are obvious:

(7.1) (i) $\mathcal{M}_s \subset \mathcal{M}_t$ for $s \leq t$;

(ii) $U_\gamma \mathcal{M}_s = \mathcal{M}_{s+\psi(\gamma)}$ for $\gamma \in \Gamma_0$;

(iii) $\bigvee_s \mathcal{M}_s = L_2(|\mu|)$: where $U_\gamma f = \chi_\gamma f$ for $\gamma \in \Gamma$ and $f \in L_2(|\mu|)$.

We want to prove

$$\bigcap_s \mathcal{M}_s = \{0\}. \tag{7.2}$$

Observe that if $f \in \mathcal{M}_{-t}$ then for γ in the set $\{\gamma_0 + u; \psi(\gamma_0) \leq t, \gamma_0 \in \Gamma_0, u \in Q\}$ we have from

$$\int_G \chi_\gamma(g)\, e(g)\, \chi_{\gamma_0+u}(g)\, |\mu|\, (dg) = \int_G \chi_{\gamma+\gamma_0+u}(g)\, \mu(dg) = 0, \tag{7.3}$$

(where the last equality follows from the fact that $\gamma + \gamma_0 + u \in Q$ since $\psi(\gamma+\gamma_0) \leq 0$ and $Q + Q \subset Q$)

that

(7.4) $$\int_G \chi_\gamma(g)\, f(g)\, \mu(dg) = 0$$

for $\gamma \in \{\gamma_0 + u;\ \psi(\gamma_0) \le t,\ \gamma_0 \in \Gamma_0,\ u \in Q\}$. If $f \in \bigcap_s \mathcal{M}_s$ then $f \in \mathcal{M}_{-t}$ for each t. Hence by (7.4) we get for all $\gamma \in \Gamma$

(7.5) $$\int_G \chi_\gamma(g)\, f(g)\, \mu(dg) = 0;$$

that is, $f(g) = 0$ a.e. μ. This implies (7.2).

Let $E(s)$ be the projection operator from $L_2(|\mu|)$ onto $\mathcal{M}_s$. Then $\{E(s),\ -\infty < s < \infty\}$ is a resolution of the identity. Let $E(\Delta)$ be the spectral measure on R generated by $\{E(s),\ -\infty < s < \infty\}$. Then for $\gamma \in \Gamma_0$, and Δ a finite subinterval of R

(7.6) $$U_\gamma\, E(\Delta)\, U_{-\gamma} = E(\Delta + \psi(\gamma)).$$

Let $V_t = \int_{-\infty}^{+\infty} e^{it\lambda}\, E(d\lambda)$. Then by Stone's theorem V_t is a (strongly continuous) group of unitary operators. From (7.6) we obtain

(7.7) $$V_t\, U_\gamma = (\psi(\gamma), t)\, U_\gamma V_t \qquad \text{for } \gamma \in \Gamma_0 \text{ and } t \in R.$$

By the property of the dual map φ, we get

(7.8) $$V_t\, U_\gamma = (\gamma, \varphi(t))\, U_\gamma\, V_t \qquad \text{for } \gamma \in \Gamma_0 \text{ and } t \in R.$$

Now U_γ has the representation (see Riesz and Sz-Nagy [16], p. 392)

$$U_\gamma = \int_G (\gamma, u)\, \alpha(du) \qquad \text{for } \gamma \in \Gamma,$$

and hence for $\gamma \in \Gamma_0$

$$U_\gamma = \int_G (\gamma, u)\, \alpha(du).$$

Since $\hat{\Gamma}_0 = G/\Gamma_0^\perp$ we get again for $\gamma \in \Gamma_0$, (7.9) $$U_\gamma = \int_{G/\Gamma_0^\perp} (\gamma,u)\ \beta(du).$$

Using the transformation $\vartheta = \rho(u)$ in (7.9) we get

(7.10) $$U_\gamma = \int_{G/\Gamma_0^\perp} (\gamma,\vartheta)\ \alpha \circ \rho^{-1}(d\vartheta) \qquad \text{for } \gamma \in \Gamma_0,$$

where $\alpha \circ \rho^{-1}(\sigma) = \alpha(\rho^{-1}(\sigma))$ for $\sigma \in \mathcal{B}(G/\Gamma_0^\perp)$. Since U_γ is is the multiplication by $\chi_\gamma(\cdot)$ we get that $\alpha(\rho^{-1}(\sigma))$ is the multiplication by $I_{\rho^{-1}(\sigma)}(\cdot)$. Thus we obtain $\beta(\sigma) = I_{\rho^{-1}(\sigma)}(\cdot)$ from (7.9) and (7.10). Now from (7.8) we have (see Theorem 2.1 of [14])

(7.11) $$V_t\ \beta(\sigma) = \beta(\sigma + \varphi(t))\ V_t.$$

Equation (7.11) and the fact that $|\mu| \circ \rho^{-1}(\sigma) = 0$ if and only if $\beta(\sigma) = 0$, yields that $|\mu| \circ \rho^{-1}(\sigma) = 0$ iff $|\mu| \circ \rho^{-1}(\sigma + \varphi(t)) = 0$. We thus have the following theorem, the form of which was suggested by Professor I. Glicksberg.

<u>Theorem 7.1</u>: Let G be a compact group and Γ its dual. Suppose that Q is a subsemigroup of Γ and Γ_0 a subgroup of Γ such that $\Gamma_0 + Q = \Gamma$. Let ψ be a homomorphism of Γ_0 to R such that $\psi(\gamma) \le 0 \Rightarrow \gamma + Q \subset Q$. Let μ be a regular Borel complex measure on G such that $\hat{\mu}(u) = 0$ for $u \in Q$. Then $|\mu| \circ \rho^{-1}$ is $\varphi(R)$-quasi-invariant where ρ is the natural homomorphism of $G \to G/\Gamma_0^\perp$ and φ is the dual homomorphism of ψ from $R \to G/\Gamma_0^\perp$.

As an application of the above result we obtain the quasi-invariance of analytic measures (see [11], [14]). Let μ be a finite complex regular measure on $\mathcal{B}(G)$. An ordering on Γ is given by a fixed non-trivial continuous homomorphism ψ of Γ into the group of real numbers R. Let φ be the dual homomorphism of ψ. μ is said to be

φ-analytic if $\int_G (\gamma,u)\, \mu(du) = 0$ when $\psi(\gamma) < 0$. Let $|\mu|$ be the total variation measure of μ. μ is called quasi-invariant if $|\mu|(\sigma) = 0$ implies $|\mu|(\sigma + \varphi(t)) = 0$ for all t in R. Putting $Q = \{\gamma | \psi(\gamma) < 0\}$ and $\Gamma_0 = \Gamma$ we obtain that ρ is the identity map and hence

Corollary 7.1: (Theorem 3.1, [14]) A φ-analytic Borel measure on $\mathcal{B}(G)$ is $\varphi(R)$-quasi-invariant.

8. Concluding Remarks.

One may find several other applications of Theorem 3.1. One such application will be to obtain the first conclusion (Part (4)) of Theorem 3.1 of Forelli [ ]. We first introduce the following notation due to Forelli: We shall denote by R^k the Euclidean k-space and by R^k_+ the Cartesian product of positive real numbers. Let W be a unit vector in R^k and M a subset of R. Then M^W will denote the strip or slab of vectors R^k whose inner product with W are in M. For every vector $\nu \in R^k$, $\chi_\nu(\cdot)$ will denote the character $\chi_\nu(x) = \exp(i < \nu, x >)$.

Proposition 8.1. Let ν be a non-negative Borel measure on R^k and assume that there is a closed subspace $\mathcal{M}$ of $L_2(\nu)$ such that

(i) $\chi_\nu \mathcal{M} \subset \mathcal{M}$ for $\nu \in R^k$,

(ii) $\bigvee_R \chi_{tW} \mathcal{M} = \{0\}$ for each unit vector $W \in R^k_+$,

(iii) $\bigwedge_R \chi_{tW} \mathcal{M} = \{0\}$ for each unit vector $W \in R^k_+$.

Then, the strip M^W is a ν-null set iff M is a set of Lebesque measure zero, where M is a Borel subset of R and W is a unit vector in R^k_+.

Proof: Let W be a fixed unit vector in R^k_+ and denote by U^W_t (for each t) the unitary operator on $L_2(\nu)$ defined as $U^W_f f(\cdot) = \chi_{tW}(\cdot) f(\cdot)$ for all $f \in L_2(\nu)$. Let $E^W(a, b]$ denote the projector operator from $L_2(\nu)$ onto $\chi_{bW}\mathcal{M} \ominus \chi_{aW}\mathcal{M}$. Obviously, $E^W(a, b]$ gives a spectral measure on R because of (ii) and (iii). Let

$V_t^W = \int \exp(it\lambda)E^W(d\lambda)$. Then V_t^W is a strongly continuous group of unitary operators and $V_t^W U_s^W = \exp(its)\, U_s^W V_t^W$. If $\beta^W(M)$ denotes the spectral measure of U_t^W then we have from the above equation, $\beta^W(M)$ stationary. Furthermore by the definition of U_t^W, $\beta^W(M)f(\cdot) = I_{M^W}(\cdot)f(\cdot)$ for all $f \in L_2(\nu)$. By Theorem 3.3 there exists on invertible isometry S^W from $L_2(\nu)$ onto $\sum_1^N \oplus L_2(R, \mu)$, where μ is the Lebesque measure, such that $S^W \beta^W(M)(S^W)^{-1}\underline{f} = (I_M(\cdot)f_1(\cdot), \ldots, I_M(\cdot)f_N(\cdot))^*$ where $\underline{f} = (f_1(\cdot), \ldots, f_N(\cdot))^*$. Now $\nu(M^W) = 0$ iff the operator $\beta^W(M) \equiv 0$ iff the operator given by the multiplication by $I_M(\cdot)$ is zero on $\sum_1^N \oplus L_2(R, \mu)$ iff $\mu(M) = 0$.

However, it is not clear that the second conclusion of the same theorem falls in our setting; apparently, a study of a family of stationary spectral measures is required to obtain it.

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